

UNIVERSAL R -MATRIX OF QUANTUM AFFINE $\mathfrak{gl}(1, 1)$

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ABSTRACT. The universal R -matrix of the quantum affine superalgebra associated to the Lie superalgebra $\mathfrak{gl}(1, 1)$ is realized as the Casimir element of certain Hopf pairing, based on the explicit coproduct formula of all the Drinfeld loop generators.

1. INTRODUCTION

Let q be a non-zero complex number which is not a root of unity. Let $\mathfrak{g} := \mathfrak{gl}(1, 1)$ be the simplest general linear Lie superalgebra. Let $U = U_q(\widehat{\mathfrak{g}})$ be the associated RTT-type quantum affine superalgebra without central charge and without derivation. As a deformation of the universal enveloping algebra of the loop Lie superalgebra $\mathfrak{g}[t, t^{-1}]$, this is a Hopf superalgebra neither commutative nor cocommutative. In this paper we study its quasi-triangular structure, namely we compute its *universal R -matrix*, an invertible element in a completed tensor product $U \widehat{\otimes} U$ satisfying notably (Remark 5.1)

$$\mathcal{R}\Delta(x) = \Delta^{\text{cop}}(x)\mathcal{R} \quad \text{for } x \in U.$$

In the non-graded case, the universal R -matrix for quantum affine algebras (non-twisted and twisted) has been obtained by Khoroshkin-Tolstoy [KT2] and Damiani [Da1, Da2]. It plays a fundamental rôle in the representation theory of quantum affine algebras: it was used by Frenkel-Reshetikhin [FR] to define the notion of q -character of a finite-dimensional representation, by Frenkel-Hernandez [FH] on Baxter polynomiality of spectra of quantum integrable systems associated to quantum affine algebras, and by Kashiwara et al. on a cyclicity property [Ka] of tensor products of finite-dimensional simple representations and on generalized quantum affine Schur-Weyl duality [KKK], to name a few.

In a series of papers [Zh1, Zh2, Zh3] devoted to the study of quantum affine superalgebras associated with the Lie superalgebras $\mathfrak{gl}(M, N)$, the highest ℓ -weight classification of finite-dimensional simple representations, a cyclicity result of tensor products of fundamental representations and q -characters were obtained. These results are similar (sometimes simpler) to the non-graded case of quantum affine algebras. We would like to look for explicit formulas of the universal R -matrix for these quantum affine superalgebras.

Khoroshkin-Tolstoy [KT1] proposed the notion of Cartan-Weyl basis to produce the universal R -matrix of quantum groups associated with finite-dimensional contragredient Lie superalgebras. In the affine case, there are explicit formulas of the universal R -matrix for: Yangian double associated to $\mathfrak{gl}(1, 1)$ in [CWWX], to $\mathfrak{osp}(1, 2)$ in [ACFR] and to $\mathfrak{gl}(M, N)$ in [RS]; quantum affine superalgebra associated to $\mathfrak{gl}(2, 2)$ in [Ga] and to $C_q^{(2)}(2)$ in [IZ].

In the present paper, we treat the special case $\mathfrak{gl}(1, 1)$. As indicated in [Zh2], the quantum affine superalgebra U is the Drinfeld quantum double of a Hopf pairing $\varphi : A \times B \rightarrow \mathbb{C}$,

where A, B are upper and lower Borel subalgebras respectively. We prove that φ is non-degenerate by exhibiting orthonormal bases for A and B with respect to φ ; these bases are formed of ordered products of Drinfeld generators (up to scalar). The universal R -matrix of U is the Casimir element for φ ; see Equations (5.12)-(5.16) for precise formulas.

The arguments in this paper is similar to [Da1] and simpler; ordered products of Drinfeld generators are already orthogonal, which is not true in the non-graded case for the Cartan loop generators. In [CWWX], for the closely related Yangian double $DY(\mathfrak{gl}(1, 1))$, universal R -matrix was written down (without proof) as the Casimir element of a certain Hopf pairing following [KT3] relating Drinfeld-Jimbo coproduct to Drinfeld new coproduct by a *twist* (see also the end of §5). Our approach does not need any twist, and is more transparent thanks to the nice coproduct formula of Drinfeld generators. The universal R -matrix of U looks simpler and is easy to specialize to certain representations.

The general case $\mathfrak{gl}(M, N)$ is still under investigation: analogous Hopf pairing φ exists; the coproduct formulas for Drinfeld generators are much more complicated (even for \mathfrak{gl}_2). It is question to find similar orthonormal bases for the underlying Hopf pairing.

The plan of this paper is as follows. §2 proves the coproduct formula for all the Drinfeld generators. §3 computes the Hopf pairing φ in terms of Drinfeld generators. §4 proves some orthogonal properties of the Hopf pairing, resulting in the universal R -matrix written down in §5. As illustrating examples, the Perk-Schultz R -matrix, which is used to define the RTT-type quantum affine superalgebra U , comes from a specialization of the universal R -matrix on natural representations; Baxter polynomiality of transfer matrices on tensor products of finite-dimensional simple representations is deduced in the spirit of Frenkel-Hernandez; Drinfeld new coproduct is realized as a twist of the RTT coproduct.

2. HOPF SUPERALGEBRA STRUCTURE

In this section, based on the Gauss decomposition, we write down the commuting relations among the Drinfeld generators of the quantum affine superalgebra associated to $\mathfrak{gl}(1, 1)$. Furthermore, we compute the coproduct for all these Drinfeld generators.

Let $\mathbf{V} := \mathbb{C}v_1 \oplus \mathbb{C}v_2$ be the vector superspace with the \mathbb{Z}_2 -grading:

$$|v_1| = |1| := \bar{0}, \quad |v_2| = |2| := \bar{1}.$$

Set $d_1 := 1, q_1 := q, d_2 := -1$ and $q_2 := q^{-1}$. Recall the *Perk-Schultz R -matrix*

$$(2.1) \quad \begin{aligned} R(z, w) &= \sum_{i=1}^2 (zq_i - wq_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\ &\quad + z(q - q^{-1}) E_{21} \otimes E_{12} + w(q^{-1} - q) E_{12} \otimes E_{21}. \end{aligned}$$

Here the $E_{ij} \in \text{End} \mathbf{V}$ for $i, j = 1, 2$ are the linear endomorphisms: $E_{ij}(v_k) = \delta_{jk} v_i$.

The quantum affine superalgebra $U := U_q(\widehat{\mathfrak{gl}(1, 1)})$ is the superalgebra defined by the RTT-generators $s_{ij}^{(n)}, t_{ij}^{(n)}$ for $i, j = 1, 2$ and $n \in \mathbb{Z}_{\geq 0}$, with the \mathbb{Z}_2 -grading $|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i| + |j|$, and with the following RTT-relations

$$\begin{aligned} R_{23}(z, w) T_{12}(z) T_{13}(w) &= T_{13}(w) T_{12}(z) R_{23}(z, w) \in (U \otimes \text{End} \mathbf{V}^{\otimes 2})((z^{-1}, w^{-1})), \\ R_{23}(z, w) S_{12}(z) S_{13}(w) &= S_{13}(w) S_{12}(z) R_{23}(z, w) \in (U \otimes \text{End} \mathbf{V}^{\otimes 2})[[z, w]], \\ R_{23}(z, w) T_{12}(z) S_{13}(w) &= S_{13}(w) T_{12}(z) R_{23}(z, w) \in (U \otimes \text{End} \mathbf{V}^{\otimes 2})((z^{-1}, w)), \end{aligned}$$

$$t_{12}^{(0)} = s_{21}^{(0)} = 0, \quad t_{ii}^{(0)} s_{ii}^{(0)} = 1 = s_{ii}^{(0)} t_{ii}^{(0)} \quad \text{for } i = 1, 2.$$

Here $T(z) = \sum_{i,j=1}^2 t_{ij}(z) \otimes E_{ij} \in (U \otimes \text{End} \mathbf{V})[[z^{-1}]]$ and $t_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} t_{ij}^{(n)} z^{-n} \in U[[z^{-1}]]$ (similar convention for $S(z)$ with the z^{-n} replaced by the z^n).

U is a Hopf superalgebra with coproduct (set $\epsilon(i, j, k) := (-1)^{(|i|+|k|)(|k|+|j|)}$)

$$\Delta(s_{ij}(z)) = \sum_{k=1}^2 \epsilon(i, j, k) s_{ik}(z) \otimes s_{kj}(z), \quad \Delta(t_{ij}(z)) = \sum_{k=1}^2 \epsilon(i, j, k) t_{ik}(z) \otimes t_{kj}(z).$$

It is \mathbb{Z} -graded in such a way that $|s_{ij}^{(n)}|_{\mathbb{Z}} = n = -|t_{ij}^{(n)}|_{\mathbb{Z}}$. Introduce

$$\mathbf{P} := \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2, \quad \alpha := \epsilon_1 - \epsilon_2, \quad \mathbf{Q} := \mathbb{Z}\alpha, \quad \mathbf{Q}_{\geq 0} := \mathbb{Z}_{\geq 0}\alpha.$$

Let $(,) : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{Z}$ be the bilinear form $(\epsilon_i, \epsilon_j) = \delta_{ij} d_i$. Then U is \mathbf{Q} -graded by setting: $x \in (U)_{\beta}$ if $s_{ii}^{(0)} x (s_{ii}^{(0)})^{-1} = q^{(\beta, \epsilon_i)} x$ for $i = 1, 2$. In particular, $|s_{ij}^{(n)}|_{\mathbf{Q}} = |t_{ij}^{(n)}|_{\mathbf{Q}} = \epsilon_i - \epsilon_j$. We remark that the \mathbb{Z} -grading and the \mathbf{Q} -grading respect the Hopf superalgebra structure.

The superalgebra U admits another system of generators, the *Drinfeld generators*:

$$E_n, \quad F_n, \quad h_s, \quad C_s, \quad (s_{ii}^{(0)})^{\pm 1}, \quad \text{for } n \in \mathbb{Z}, \quad s \in \mathbb{Z}_{\neq 0} \text{ and } i = 1, 2.$$

The commuting relations among these generators are as follows:

$$(D1) \quad |(s_{ii}^{(0)})^{\pm 1}|_{\mathbf{Q}} = |h_s|_{\mathbf{Q}} = |C_s|_{\mathbf{Q}} = 0 \text{ and } |E_n|_{\mathbf{Q}} = \alpha = -|F_n|_{\mathbf{Q}};$$

$$(D2) \quad \text{the } C_s \text{ are central elements and } [h_s, h_t] = 0 \text{ for } s, t \in \mathbb{Z}_{\neq 0};$$

$$(D3) \quad \text{for } n \in \mathbb{Z} \text{ and } s \in \mathbb{Z}_{\neq 0} \text{ we have (denote } [s] := \frac{q^s - q^{-s}}{q - q^{-1}})$$

$$[h_s, E_n] = q^s \frac{[s]}{s} E_{n+s}, \quad [h_s, F_n] = -q^s \frac{[s]}{s} F_{n+s};$$

$$(D4) \quad [E_m, F_n] = (q - q^{-1})(\phi_{m+n}^+ - \phi_{m+n}^-) \text{ where the } \phi_n^{\pm} \text{ are defined by}$$

$$\phi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \phi_n^{\pm} z^n := ((s_{11}^{(0)})^{-1} s_{22}^{(0)})^{\pm 1} \exp(\pm(q - q^{-1}) \sum_{s>0} C_{\pm s} z^{\pm s}) \in U[[z^{\pm 1}]];$$

$$(D5) \quad [E_m, E_n] = [F_m, F_n] = 0 \text{ for } m, n \in \mathbb{Z}.$$

Introduce the following formal series with coefficients in U :

$$E^+(z) = \sum_{n \geq 0} E_n z^n, \quad E^-(z) = - \sum_{n \leq -1} E_n z^n, \quad F^+(z) = - \sum_{n \geq 1} F_n z^n, \quad F^-(z) = \sum_{n \leq 0} F_n z^n,$$

$$K_1^{\pm}(z) = (s_{11}^{(0)})^{\pm 1} \exp(\pm(q - q^{-1}) \sum_{s>0} h_{\pm s} z^{\pm s}), \quad K_2^{\pm}(z) = K_1^{\pm}(z) \phi^{\pm}(z).$$

The Drinfeld and RTT generators are related to each other by the Gauss decomposition:

$$\begin{pmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F^+(z) & 1 \end{pmatrix} \begin{pmatrix} K_1^+(z) & 0 \\ 0 & K_2^+(z) \end{pmatrix} \begin{pmatrix} 1 & E^+(z) \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F^-(z) & 1 \end{pmatrix} \begin{pmatrix} K_1^-(z) & 0 \\ 0 & K_2^-(z) \end{pmatrix} \begin{pmatrix} 1 & E^-(z) \\ 0 & 1 \end{pmatrix}$$

as matrix equations over the superalgebras $U[[z]]$ and $U[[z^{-1}]]$ respectively. Note that the sub-index for the E, F, C, h, ϕ^{\pm} refers to the \mathbb{Z} -degree.

The main result of this section is the coproduct formulas for *all* the Drinfeld generators.

Proposition 2.1. *The coproduct of Drinfeld generators is as follows:*

$$(2.2) \quad \Delta(E^\pm(z)) = 1 \otimes E^\pm(z) + E^\pm(z) \otimes \phi^\pm(z),$$

$$(2.3) \quad \Delta(F^\pm(z)) = \phi^\pm(z) \otimes F^\pm(z) + F^\pm(z) \otimes 1,$$

$$(2.4) \quad \Delta(\phi^\pm(z)) = \phi^\pm(z) \otimes \phi^\pm(z), \quad \Delta(C_s) = 1 \otimes C_s + C_s \otimes 1 \quad \text{for } s \in \mathbb{Z}_{\neq 0},$$

$$(2.5) \quad \Delta(h_s) = 1 \otimes h_s + h_s \otimes 1 + \frac{q^s[s]}{s(q - q^{-1})} \sum_{i=0}^{s-1} E_i \otimes F_{s-i} \quad \text{for } s > 0,$$

$$(2.6) \quad \Delta(h_{-s}) = 1 \otimes h_{-s} + h_{-s} \otimes 1 + \frac{q^{-s}[s]}{s(q^{-1} - q)} \sum_{i=0}^{s-1} E_{-s+i} \otimes F_{-i} \quad \text{for } s > 0.$$

Proof. Equations (2.2)-(2.4) were proved in [CWWZ]. As the idea is simple, for completeness we give a proof to the formulas $\Delta(E_n)$ with $n \geq 0$. From the Gauss decomposition:

$$\Delta(h_1) = 1 \otimes h_1 + h_1 \otimes 1 + \frac{q}{q - q^{-1}} E_0 \otimes F_1, \quad \Delta(E_0) = 1 \otimes E_0 + E_0 \otimes \phi_0^+.$$

Assume the formula $\Delta(E_n)$. Since $[h_1, E_n] = qE_{n+1}$, we have

$$\begin{aligned} \Delta(qE_{n+1}) &= [1 \otimes h_1 + h_1 \otimes 1 + \frac{q}{q - q^{-1}} E_0 \otimes F_1, 1 \otimes E_n + \sum_{i=0}^n E_i \otimes \phi_{n-i}^+] \\ &= 1 \otimes qE_{n+1} + \sum_{i=0}^n qE_{i+1} \otimes \phi_{n-i}^+ + \frac{q}{q - q^{-1}} E_0 \otimes [F_1, E_n] \\ &= q(1 \otimes E_{n+1} + \sum_{i=0}^{n+1} E_i \otimes \phi_{n+1-i}^+), \end{aligned}$$

giving the desired formula for $\Delta(E_{n+1})$.

Let us prove Equation (2.5). (The idea applies perfectly well to Equation (2.6)!) Introduce the following power series with coefficients in $U^{\otimes 2}$

$$\begin{aligned} \mu(z) &:= \sum_{s>0} (q - q^{-1})(1 \otimes h_s + h_s \otimes 1 + \frac{q^s[s]}{s(q - q^{-1})} \sum_{i=0}^{s-1} E_i \otimes F_{s-i}) z^s =: \sum_{s>0} \mu_s z^s, \\ \tilde{\mu}(z) &:= \exp((q - q^{-1}) \sum_{s>0} \Delta(h_s) z^s), \quad \mu'(z) := \frac{d}{dz} \mu(z). \end{aligned}$$

Claim 1. For $s, t \in \mathbb{Z}_{>0}$, $\mu_s \mu_t = \mu_t \mu_s \in U^{\otimes 2}$.

This follows from a straightforward calculation. Equation (2.5) then becomes

$$(a) : \quad \frac{d}{dz} \tilde{\mu}(z) = \tilde{\mu}(z) \mu'(z) \in U^{\otimes 2}[[z]].$$

Set $H(z) := \sum_{s>0} (q - q^{-1}) h_s z^s \in U[[z]]$. By definition, we have

$$\tilde{\mu}(z) = e^{H(z)} \otimes e^{H(z)} - q e^{H(z)} E^+(z) \otimes F^+(z) e^{H(z)}.$$

Now (a) left multiplied by $e^{-H(z)} \otimes 1$ and right multiplied by $q^{-1} \otimes e^{-H(z)}$ becomes:

$$([\frac{dH(z)}{dz}, E^+(z)] + \frac{dE^+(z)}{dz}) \otimes F^+(z) + E^+(z) \otimes \frac{dF^+(z)}{dz} = (E^+(z) \otimes F^+(z) - q^{-1})x(z).$$

Here $x(z) := \sum_{s>0} z^{s-1} q^s [s] \sum_{i=0}^{s-1} E_i \otimes e^{H(z)} F_{s-i} e^{-H(z)}$. By Relations (D2)-(D3)

$$e^{H(z)} F_n e^{-H(z)} = \sum_{m \geq 0} c_m F_{n+m} z^m \quad \text{with the } c_m \text{ defined by}$$

$$\sum_{m \geq 0} c_m z^m := \exp((q - q^{-1}) \sum_{s>0} \frac{-q^s [s]}{s} z^s) = \exp(\sum_{s>0} \frac{1 - q^{2s}}{s} z^s) = \frac{1 - q^2 z}{1 - z}.$$

It follows that

$$\begin{aligned} x(z) &= \sum_{s>0} z^{s-1} q^s [s] \sum_{i=0}^{s-1} \sum_{m \geq 0} E_i \otimes F_{m+s-i} c_m z^m = \sum_{l \geq 0} z^l \sum_{i=0}^l x_{i,l} E_i \otimes F_{l+1-i}, \\ x_{i,l} &= \sum_{s=i+1}^{l+1} c_{l+1-s} q^s [s] = q^{i+1} [i+1] + q(l-i) \quad \text{for } 0 \leq i \leq l. \end{aligned}$$

Claim 2. We have $(E^+(z) \otimes F^+(z))x(z) = 0 \in U^{\otimes 2}[[z]]$.

Let $0 \leq j < k$ and $1 \leq a < b$. Consider the term $E_j E_k \otimes F_a F_b$ appearing at the LHS:

$$\begin{aligned} &(E_j \otimes F_a)(E_k \otimes F_b)x_{k,b+k-1} z^{a+b+j+k-1} + (E_j \otimes F_b)(E_k \otimes F_a)x_{k,a+k-1} z^{a+b+j+k-1} \\ &+ (E_k \otimes F_a)(E_j \otimes F_b)x_{j,b+j-1} z^{a+b+j+k-1} + (E_k \otimes F_b)(E_j \otimes F_a)x_{j,a+j-1} z^{a+b+j+k-1} \\ &= E_j E_k \otimes F_a F_b z^{a+b+j+k-1} (-x_{k,b+k-1} + x_{k,a+k-1} + x_{j,b+j-1} - x_{j,a+j-1}). \end{aligned}$$

It is straightforward to check that $-x_{k,b+k-1} + x_{k,a+k-1} + x_{j,b+j-1} - x_{j,a+j-1} = 0$. Claim 2 therefore follows. Equation (a) becomes:

$$(b) : [\frac{dH(z)}{dz}, E^+(z)] \otimes F^+(z) + \frac{dE^+(z)}{dz} \otimes F^+(z) + E^+(z) \otimes \frac{dF^+(z)}{dz} = -q^{-1}x(z).$$

By using $[h_s, E_n] = \frac{q^s [s]}{s} E_{n+s}$ we get (after direct calculations)

$$[\frac{dH(z)}{dz}, E^+(z)] + \frac{dE^+(z)}{dz} = \sum_{a \geq 0} q^{a+2} [a+1] E_{a+1} z^a.$$

Now Equation (b) follows from: $q^{a+1} [a] + b = q^{-1} x_{a,a+b-1}$ for $a \geq 0, b > 0$. \square

Compared to [CWWZ], Equations (2.5)-(2.6) are simpler as the Cartan loop generators h_s are chosen differently, which will simplify the universal R -matrix in §5.

3. QUANTUM DOUBLE

In this section, we recall the quantum double construction of U and compute explicitly the associated Hopf pairing in terms of Drinfeld generators.

Let A (resp. B) be the subalgebra of U generated by the $s_{ij}^{(n)}, t_{ii}^{(0)}$ (resp. the $t_{ij}^{(n)}, s_{ii}^{(0)}$). Then A and B are sub-Hopf-superalgebras of U . In terms of Drinfeld generators, A

(resp. B) is generated by the $(s_{ii}^{(0)})^{-1}$ (resp. the $(t_{ii}^{(0)})^{-1}$) and the coefficients of the $E^+(z), F^+(z), K_1^+(z), \phi^+(z)$ (resp. with $+$ replaced by $-$).

There exists a Hopf pairing $\varphi : A \times B \longrightarrow \mathbb{C}$, which is an even bilinear form satisfying

$$\begin{aligned}\varphi(a, bb') &= (-1)^{|b||b'|} \varphi(a_{(1)}, b) \varphi(a_{(2)}, b'), \quad \varphi(a, 1) = \varepsilon(a); \\ \varphi(aa', b) &= \varphi(a', b_{(1)}) \varphi(a, b_{(2)}), \quad \varphi(1, b) = \varepsilon(b)\end{aligned}$$

for homogeneous $a, a' \in A$ and $b, b' \in B$. φ is determined by [Zh2, Proposition 3.10]

$$(3.7) \quad \sum_{i,j,a,b=1}^2 \varphi(s_{ij}(w), t_{ab}(z)) E_{ab} \otimes E_{ij} = \frac{R(z, w)}{zq - wq^{-1}} \in \text{End} \mathbf{V}^{\otimes 2} \left[\left[\frac{w}{z} \right] \right].$$

φ makes the tensor product $A \otimes B$ into a Hopf superalgebra $\mathcal{D}_\varphi(A, B)$, called *quantum double*. U is the quotient of $\mathcal{D}_\varphi(A, B)$ by the Hopf ideal generated by the $s_{ii}^{(0)} \otimes 1 - 1 \otimes s_{ii}^{(0)}$.

We remark that $\varphi : A \times B \longrightarrow \mathbb{C}$ respects the \mathbb{Z} -grading and the \mathbf{Q} -grading. In other words, let $\beta, \gamma \in \mathbb{Z}$ or \mathbf{Q} and let $x \in (A)_\beta, y \in (B)_\gamma$. Then $\varphi(x, y) = 0$ if $\beta + \gamma \neq 0$.

Proposition 3.1. *The Hopf pairing $\varphi : A \times B \longrightarrow \mathbb{C}$ satisfies:*

$$(3.8) \quad \varphi(K_1^+(w), K_1^-(z)) = 1 = \varphi(\phi^+(w), \phi^-(z)),$$

$$(3.9) \quad \varphi(E^+(w), F^-(z)) = \frac{(q - q^{-1})z}{z - w}, \quad \varphi(\phi^+(w), K_1^-(z)) = \frac{z - w}{qz - q^{-1}w},$$

$$(3.10) \quad \varphi(F^+(w), E^-(z)) = \frac{(q^{-1} - q)w}{z - w}, \quad \varphi(K_1^+(w), \phi^-(z)) = \frac{q^{-1}z - qw}{z - w}.$$

Proof. As $K_1^+(w) = s_{11}(w)$ and $K_1^-(z) = t_{11}(z)$, we have by Equation (3.7)

$$\varphi(K_1^+(w), K_1^-(z)) = \varphi(s_{11}(w), t_{11}(z)) = 1, \quad \varphi(s_{ii}^{(0)}, t_{jj}^{(0)}) = q^{-1+(\epsilon_i, \epsilon_j)} \quad \text{for } i, j = 1, 2.$$

It follows that $\varphi(\phi_0^+, \phi_0^-) = \varphi((s_{11}^{(0)})^{-1} s_{22}^{(0)}, (t_{11}^{(0)})^{-1} t_{22}^{(0)}) = 1$. More generally, $\varphi(\phi_m^+, \phi_{-n}^-) = \delta_{m,0} \delta_{n,0}$ in view of Equation (2.4) and the compatibility of \mathbf{Q} -grading with φ .

Consider now Equation (3.9). Let us set

$$\varphi(\phi^+(w), K_1^-(z)) = g\left(\frac{w}{z}\right), \quad \varphi(E^+(w), F^-(z)) = h\left(\frac{w}{z}\right).$$

In Equation (3.7), by taking $(i, j, a, b) = (1, 2, 2, 1)$ we get

$$(a) : \quad \varphi(K_1^+(w) E^+(w), F^-(z) K_1^-(z)) = \varphi(s_{12}(w), t_{21}(z)) = \frac{(q - q^{-1})z}{qz - q^{-1}w}.$$

The first equality comes from the Gauss decomposition. From the coproduct formula

$$\begin{aligned}\Delta(K_1^+(w) E^+(w)) &= K_1^+(w) E^+(w) \otimes K_1^+(w) \phi^+(w) - s_{12}(w) \otimes s_{21}(w) E^+(w) \\ &\quad + K_1^+(w) \otimes K_1^+(w) E^+(w) + s_{12}(w) E^+(w) \otimes s_{21}(w) \phi^+(w)\end{aligned}$$

it follows that the LHS of Equation (a) takes the form

$$\begin{aligned}LHS(a) &= \varphi(K_1^+(w) E^+(w), F^-(z)) \varphi(K_1^+(w) \phi^+(w), K_1^-(z)) \\ &\quad - \varphi(s_{12}(w), F^-(z)) \varphi(s_{21}(w) E^+(w), K_1^-(z))\end{aligned}$$

The coproduct formula for $K_1^-(z), F^-(z)$ implies that $\varphi(s_{21}(w)E^+(w), K_1^-(z)) = 0$ and

$$\begin{aligned}\varphi(K_1^+(w)\phi^+(w), K_1^-(z)) &= \varphi(\phi^+(w), K_1^-(z))\varphi(K_1^+(w), K_1^-(z)) = g\left(\frac{w}{z}\right), \\ \varphi(K_1^+(w)E^+(w), F^-(z)) &= \varphi(E^+(w), F^-(z))\varphi(K_1^+(w), 1) = h\left(\frac{w}{z}\right).\end{aligned}$$

Henceforth Equation (a) gives rise to the following equation:

$$(b) : \quad g\left(\frac{w}{z}\right)h\left(\frac{w}{z}\right) = \frac{(q - q^{-1})z}{qz - q^{-1}w}.$$

Let us compute $g\left(\frac{w}{z}\right)$ in a different way by using

$$\phi^+(w) = \phi_0^+ + \frac{w}{q - q^{-1}}(E^+(w)F_1 + F_1E^+(w)).$$

The coproduct formula for $K_1^-(z)$ implies that $\varphi(F_1E^+(w), K_1^-(z)) = 0$ and

$$\begin{aligned}g\left(\frac{w}{z}\right) &= \varphi(\phi_0^+, K_1^-(z)) + \frac{w}{q - q^{-1}}\varphi(E^+(w)F_1, K_1^-(z)) \\ &= q^{-1} + \frac{w}{q^{-1} - q}\varphi(F_1, t_{12}(z))\varphi(E^+(w), t_{21}(z)) \\ &= q^{-1} + \frac{w}{q^{-1} - q}\varphi(F_1, t_{12}^{(1)}z^{-1})\varphi(E^+(w), F^-(z)K_1^-(z)).\end{aligned}$$

From the coproduct formula for $E^+(w)$ and from $F_1 = -s_{21}^{(1)}(s_{11}^{(0)})^{-1}$,

$$\begin{aligned}\varphi(E^+(w), F^-(z)K_1^-(z)) &= \varphi(E^+(w), F^-(z))\varphi(\phi^+(w), K_1^-(z)) = g\left(\frac{w}{z}\right)h\left(\frac{w}{z}\right) = \frac{(q - q^{-1})z}{qz - q^{-1}w}, \\ \varphi(F_1, t_{12}^{(1)}) &= -\varphi(s_{21}^{(1)}(s_{11}^{(0)})^{-1}, t_{21}^{(1)}) = -\varphi((s_{11}^{(0)})^{-1}, t_{11}^{(0)})\varphi(s_{21}^{(1)}, t_{21}^{(1)}) = -\frac{q^{-1} - q}{q}, \\ g\left(\frac{w}{z}\right) &= q^{-1} - \frac{w}{z} \frac{1}{q^{-1} - q} \frac{q^{-1} - q}{q} \frac{(q - q^{-1})z}{qz - q^{-1}w} = \frac{z - w}{qz - q^{-1}w}.\end{aligned}$$

Now Equation (b) gives us the desired formula for $h\left(\frac{w}{z}\right)$. Equation (3.9) is proved.

The proof of Equation (3.10) is parallel to that of Equation (3.9). \square

In terms of Drinfeld generators, Proposition 3.1 becomes

Corollary 3.2. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $s, t \in \mathbb{Z}_{> 0}$. We have*

$$\begin{aligned}\varphi(h_s, h_{-t}) &= \varphi(C_s, C_{-t}) = 0, \quad \varphi(h_s, C_{-t}) = \delta_{st} \frac{q^s[s]}{s(q - q^{-1})}, \quad \varphi(C_s, h_{-t}) = \delta_{st} \frac{q^{-s}[s]}{s(q - q^{-1})}, \\ \varphi(E_m, F_{-n}) &= \delta_{mn}(q - q^{-1}), \quad \varphi(F_s, E_{-t}) = \delta_{st}(q^{-1} - q).\end{aligned}$$

There are other choices of Hopf pairing (for example, by replacing the denominator at the RHS of Equation (3.7) with $zq^{-1} - wq$ or more generally with $zx - wy$ where $x, y \in \mathbb{C}^\times$), which should lead to slightly different universal R -matrices.

4. ORTHOGONAL PBW BASES

In this section, we prove some orthogonal properties of the Hopf pairing $\varphi : A \times B \longrightarrow \mathbb{C}$. Let \mathcal{B} be the following totally ordered subset of A :

$$\begin{aligned} & \cdots < (\phi_0^+)^{-1}F_n < (\phi_0^+)^{-1}F_{n-1} < \cdots < (\phi_0^+)^{-1}F_3 < (\phi_0^+)^{-1}F_2 < (\phi_0^+)^{-1}F_1 \\ & < h_1 < h_2 < h_3 < \cdots < h_s < h_{s+1} < \cdots < C_1 < C_2 < C_3 \cdots < C_s < C_{s+1} < \cdots \\ & < E_0 < E_1 < E_2 < \cdots < E_n < E_{n+1} < \cdots \end{aligned}$$

Let $\mathcal{B}_-, \mathcal{B}_0$ and \mathcal{B}_+ be the totally ordered subset of \mathcal{B} consisting of elements from the first, second and third row above respectively. We remark that the above vectors are linearly independent, taking into account the \mathbb{Z} -grading, the \mathbf{Q} -grading and Corollary 3.2.

For $b \in \mathcal{B}$, define $b^- \in B$ in the following way: let $s \in \mathbb{Z}_{>0}, n \in \mathbb{Z}_{\geq 0}$

$$((\phi_0^+)^{-1}F_s)^- := (\phi_0^-)^{-1}E_{-s}, \quad h_s^- = C_{-s}, \quad C_s^- := h_{-s}, \quad E_n^- := F_{-n}.$$

Let Γ be the set of functions $f : \mathcal{B} \longrightarrow \mathbb{Z}_{\geq 0}$ such that: $f(b) = 0$ except for finitely many $b \in \mathcal{B}$; $f(b) \leq 1$ for $b \in \mathcal{B}_- \cup \mathcal{B}_+$. For $f \in \Gamma$, define the following ordered products:

$$E(f) := \prod_{b \in \mathcal{B}}^{\rightarrow} b^{f(b)} \in A, \quad F(f) := \prod_{b \in \mathcal{B}}^{\rightarrow} (b^-)^{f(b)} \in B.$$

Proposition 4.1. For $f, g \in \Gamma$ and $k, k' \in A \cap B$ products of the $(s_{ii}^{(0)})^{\pm 1}$,

$$(4.11) \quad \varphi(kE(f), k'F(g)) = \varphi(k, k') \delta_{f,g} (-1)^{\sum_{b < b'} f(b)f(b')|b||b'|} \prod_{b \in \mathcal{B}} f(b)! \varphi(b, b^-)^{f(b)}.$$

Proof. The idea of proof is similar to that of Damiani [Da1, Proposition 10.5]. Our situation is much more transparent as we have the explicit coproduct formula. Let us first prove the formula $\varphi(kE(f), k'F(g)) = \varphi(k, k') \varphi(E(f), F(g))$.

Let $\varphi_2 : A^{\otimes 2} \times B^{\otimes 2} \longrightarrow \mathbb{C}$ be the bilinear form

$$\varphi_2(a \otimes a', b \otimes b') := (-1)^{|b||b'|} \varphi(a, b) \varphi(a', b')$$

for a, a', b, b' homogeneous. Then from the definition of Hopf pairing

$$\varphi(kE(f), k'F(g)) = \varphi_2((k \otimes k) \Delta(E(f)), k' \otimes F(g)).$$

Claim 1. For $b \in \mathcal{B}$, we have $\Delta(b) - 1 \otimes b \in \mathcal{B}A \otimes A$.

This comes from Proposition 2.1. Since $\varphi(k\mathcal{B}A, k') = 0$, we see that $\varphi(kE(f), k'F(g)) = \varphi(k, k') \varphi(kE(f), F(g))$. Let $(\mathcal{B})^- := \{b^- \in B | b \in \mathcal{B}\}$. The following is clear.

Claim 2. For $b \in \mathcal{B}$, we have $\Delta(b^-) - b^- \otimes 1 \in B \otimes B(\mathcal{B})^-$.

Thus $\varphi(kE(f), F(g)) = \varphi(E(f), F(g))$. To prove Equation (4.11), we can assume $k = k' = 1$ and $\varphi(E(f), F(g)) \neq 0$. We proceed by induction on the *length* of $g \in \Gamma$ defined by

$$\ell(g) := \sum_{b \in \mathcal{B}} f(b).$$

If $\ell(g) = 0$, then $F(g) = 1$. Clearly, $\varphi(E(f), 1) \neq 0$ if and only if $E(f) = 1$, if and only if $f = g$. The initial statement is proved. Let $\ell(g) > 0$ and assume (4.11) whenever the length of the second function is less than $\ell(g)$. Let b_1 (resp. b_2) be the minimal (resp. maximal) element $b \in \mathcal{B}$ such that $f(b) > 0$. We consider three cases separately.

Case I: $b_1 = (\phi_0^+)^{-1}F_l \in \mathcal{B}_-$ for some $l > 0$. Let $g_1 \in \Gamma$ be obtained from g by replacing $g(b_1) = 1$ with $g_1(b_1) = 0$. By definition $F(g) = b_1^- F(g_1)$.

Claim 3. We have $\varphi(A(\mathcal{B}_0 \cup \mathcal{B}_+), b_1^-) = 0$.

This comes from the coproduct formula $\Delta(b_1^-)$; the first tensor factors of $\Delta((\phi_0^-)^{-1}E_{-l})$ are always orthogonal to $\mathcal{B}_0 \cup \mathcal{B}_+$ with respect to φ . Let us write $E(f)$ explicitly:

$$E(f) = (\phi_0^+)^{-1}F_{n_s} \cdots (\phi_0^+)^{-1}F_{n_2}(\phi_0^+)^{-1}F_{n_1} \prod_{b \in \mathcal{B}_0 \cup \mathcal{B}_+}^{\rightarrow} b^{f(b)}.$$

with $1 \leq n_1 < n_2 < \cdots < n_s$ and $s \geq 0$.

Claim 4. For $b \in \mathcal{B}_0 \cup \mathcal{B}_+$ we have $\Delta(b) - 1 \otimes b \in (\mathcal{B}_0 \cup \mathcal{B}_+) \otimes A$.

This comes from Proposition 2.1. In view of Claim 3, we see that

$$(*) : \quad \varphi(E(f), F(g)) = \varphi_2(\Delta(\prod_{i=s}^1 (\phi_0^+)^{-1}F_{n_i})(1 \otimes \prod_{b \in \mathcal{B}_0 \cup \mathcal{B}_+}^{\rightarrow} b^{f(b)}), (\phi_0^-)^{-1}E_{-l} \otimes F(g_1)).$$

From the coproduct formula $\Delta(b)$ with $b \in \mathcal{B}_-$ and from the compatibility of φ with the \mathbf{Q} -grading we deduce that $s > 0$ and there exists $1 \leq j \leq s$ with $n_j = l$. Since the $(\phi_0^+)^{-1}\phi_i^+$ for $i > 0$ are products of the central elements $C_p \in \mathcal{B}_0$, Claim 3 implies that the only possible tensor factor of $\Delta(\prod_{i=s}^1 (\phi_0^+)^{-1}F_{n_i})$ at the RHS of the above identity contributing to non-zero terms will be

$$\prod_{i=s}^{j+1} (1 \otimes (\phi_0^+)^{-1}F_{n_i}) \times ((\phi_0^+)^{-1}F_{n_j} \otimes (\phi_0^+)^{-1}) \times \prod_{i=j-1}^1 (1 \otimes (\phi_0^+)^{-1}F_{n_i}).$$

This says that the RHS of Equation (*) takes the form

$$\begin{aligned} RHS(*) &= (-1)^{s-j+|F(g')|} \varphi((\phi_0^+)^{-1}F_l, (\phi_0^-)^{-1}E_{-l}) \times \\ &\quad \varphi((\phi_0^+)^{-1}(\phi_0^+)^{-1}F_{n_s} \cdots (\widehat{(\phi_0^+)^{-1}F_{n_j}} \cdots (\phi_0^+)^{-1}F_{n_1} \prod_{b \in \mathcal{B}_0 \cup \mathcal{B}_+}^{\rightarrow} b^{f(b)}, F(g')). \end{aligned}$$

Let $f_1 \in \Gamma$ be obtained from f by replacing $f(b_1) = 1$ with $f_1(b_1) = 0$, then

$$\begin{aligned} F(g) &= b_1^- F(g_1), \quad E(f) = (-1)^{s-j} b_1 E(f_1), \\ \varphi(E(f), F(g)) &= (-1)^{s-j+|F(g_1)|} \varphi(b_1, b_1^-) \varphi(E(f_1), F(g_1)) \end{aligned}$$

Now $\varphi(E(f_1), F(g_1)) \neq 0$. The induction hypothesis applied to g_1 implies that $f_1 = g_1$. By the definition of g_1 , we see that $f = g$ and $j = s$.

Case II: $b_2 = E_n \in \mathcal{B}_+$ for some $n \geq 0$. Let $g_2 \in \Gamma$ be obtained from g by replacing $g(b_2) = 1$ with $g_2(b_2) = 0$. It follows that $F(g) = F(g_2)b_2^-$. Similar to Claims 3,4:

$$(3') \quad \varphi((\mathcal{B}_- \cup \mathcal{B}_0)A, b_2^-) = 0.$$

$$(4') \quad \text{For } b \in \mathcal{B}_-, \text{ we have } \Delta(b) - b \otimes (\phi_0^+)^{-1} \in A \otimes \mathcal{B}_-; \text{ for } b \in \mathcal{B}_0, \text{ we have } \Delta(b) - b \otimes 1 \in 1 \otimes b + A \otimes \mathcal{B}_- \phi_0^+.$$

Since ϕ_0^+ is a central element, as in the case $b_1 \in \mathcal{B}_-$ we have

$$(**) : \quad \varphi(E(f), F(g)) = \varphi_2(\Delta(E(f)), F(g_2) \otimes b_2^-)$$

$$= \varphi_2((\prod_{b \in \mathcal{B}_- \cup \mathcal{B}_0}^{\rightarrow} b^{f(b)} \otimes \prod_{b \in \mathcal{B}_-}^{\rightarrow} (\phi_0^+)^{-f(b)}) \Delta(\prod_{b \in \mathcal{B}_+}^{\rightarrow} b^{f(b)}), F(g_2) \otimes F_{-n}).$$

It follows that $f(b_2) = 1$. In particular, write

$$\prod_{b \in \mathcal{B}_+}^{\rightarrow} b^{f(b)} = E_{m_1} E_{m_2} \cdots E_{m_t}$$

with $t > 0, m_j = n$ for some $1 \leq j \leq t$ and $0 \leq m_1 < m_2 < \cdots < m_t$. Since the $(\phi_0^+)^{-1} \phi_i^+$ for $i > 0$ are products of the central elements $C_p \in \mathcal{B}_0$, by (3'), the only tensor factor of $\Delta(\prod_{i=1}^t E_{m_i})$ contributing to non-zero terms in Equation (**) will be

$$\prod_{i=1}^{j-1} (E_{m_i} \otimes \phi_0^+) \times (1 \otimes E_{m_j}) \times \prod_{i=j+1}^t (E_{m_i} \otimes \phi_0^+).$$

The last term of Equation (**) becomes (ϕ_0^+) can be ignored.)

$$(-1)^{j-t+|F(g_2)|} \varphi(E_n, F_{-n}) \varphi((\prod_{b \in \mathcal{B}_- \cup \mathcal{B}_0}^{\rightarrow} b^{f(b)}) E_{m_1} E_{m_2} \cdots \widehat{E_{m_j}} \cdots E_{m_t}, F(g_2)).$$

Let $f_2 \in \Gamma$ be obtained from f by replacing $f(b_2) = 1$ with $f_2(b_2) = 0$. Then

$$\begin{aligned} E(f) &= (-1)^{j-t} E(f_2) b_2, \quad F(g) = F(g_2) b_2^-, \\ \varphi(E(f), F(g)) &= (-1)^{j-t+|F(g_2)|} \varphi(E(f_2), F(g_2)) \varphi(b_2, b_2^-). \end{aligned}$$

The induction hypothesis implies that $f_2 = g_2$. It follows that $f = g$ and $j = t$.

Case III: $b_1, b_2 \in \mathcal{B}_0$. From the coproduct formula $\Delta(b)$ with $b \in \mathcal{B}_0$, we see that any second tensor factor of $\Delta(F(g))$ is orthogonal to \mathcal{B}_- . This says that $f(b) = 0$ whenever $b \in \mathcal{B}_-$. Next, any first tensor factor of $\Delta(F(g))$ is orthogonal to \mathcal{B}_+ . So $f(b) = 0$ whenever $b \in \mathcal{B}_+$. Write $F(g) = b_1^- F(g_1)$ with g_1 defined as in the case $b_1 \in \mathcal{B}_-$. Consider

$$\varphi(E(f), F(g)) = \varphi_2(\Delta(E(f)), b_1^- \otimes F(g_1)).$$

As $E(f)$ is a product of the $b \in \mathcal{B}_0$, from Proposition 2.1 we deduce that

$$\Delta(E(f)) - \prod_{b \in \mathcal{B}_0}^{\rightarrow} (1 \otimes b + b \otimes 1) \in A\mathcal{B}_+ \otimes A\mathcal{B}_-.$$

Note that $\varphi(A\mathcal{B}_+, b_1^-) = 0$. It follows that

$$\varphi(E(f), F(g)) = \varphi_2(\prod_{b \in \mathcal{B}_0}^{\rightarrow} (1 \otimes b + b \otimes 1)^{f(b)}, b_1^- \otimes F(g_1)).$$

According to the following lemma, $f(b_1) > 0$. Let $f_3 \in \Gamma$ be obtained from f by replacing $f(b_1)$ with $f_3(b_1) = f(b_1) - 1$. Then $E(f) = b_1 E(f_3)$ and

$$\varphi(E(f), F(g)) = f(b_1) \varphi_2(b_1 \otimes E(f_3), b_1^- \otimes F(g_1)) = f(b_1) \varphi(b_1, b_1^-) \varphi(E(f_3), F(g_1)).$$

From $\varphi(E(f_3), F(g_1)) \neq 0$ it follows that $f_3 = g_1$. Henceforth $f = g$, as desired. \square

Lemma 4.2. *Let $s > 0$ and $x_1, x_2, \dots, x_s \in \mathcal{B}_0$. Let $y \in \mathcal{B}_0$. If $\varphi(x_1 x_2 \cdots x_s, y^-) \neq 0$, then $s = 1$ and $x_1 = y$.*

Proof. If $s = 1$, then according to Corollary 3.2, $x_1 = y$. Suppose $s > 1$. Set $y' := x_1 x_2 \cdots x_{s-1}$. Then $\varphi(y', 1) = 0 = \varphi(x_s, 1)$. On the other hand, by Proposition 2.1, $\Delta(y^-) - y^- \otimes 1 - 1 \otimes y^-$ is either zero or a sum of the $E_m \otimes F_n$ with $m < 0$ and $n \leq 0$. The compatibility of φ and the \mathbf{Q} -grading says that

$$\varphi(x_s, E_m) \varphi(y', F_n) = 0.$$

So $\varphi(y' x_s, y^-) = \varphi(y', y^-) \varphi(x_s, 1) + \varphi(y', 1) \varphi(x_s, y^-) = 0$, a contradiction. \square

5. UNIVERSAL R -MATRIX

In this section, we write down the explicit formula for the universal R -matrix of the quantum affine superalgebra $U = U_q(\widehat{\mathfrak{gl}(1, 1)})$, following the general argument in [Da1, §3].

Let \hbar be a formal parameter. Let us extend U to a topological Hopf superalgebra over $\mathbb{C}[[\hbar]]$ by adding primitive elements δ_1, δ_2 and by identifying

$$q = e^{\hbar}, \quad s_{ii}^{(0)} = e^{\hbar \delta_i}, \quad t_{ii}^{(0)} = e^{-\hbar \delta_i}.$$

Now A, B and the Hopf pairing $\varphi : A \times B \rightarrow \mathbb{C}$ are extended similarly to A_{\hbar}, B_{\hbar} and $\varphi : A_{\hbar} \times B_{\hbar} \rightarrow \mathbb{C}((\hbar))$. Let us first determine the $\varphi(\delta_i, \delta_j)$ with $i, j = 1, 2$. As the $\delta_i \in U_{\hbar}$ are primitive elements, the proof of Proposition 4.1 says that

$$\varphi(s_{ii}^{(0)}, t_{jj}^{(0)}) = \varphi(e^{\hbar \delta_i}, e^{-\hbar \delta_j}) = e^{-\varphi(\hbar \delta_i, \hbar \delta_j)} = q^{-\hbar \varphi(\delta_i, \delta_j)}.$$

It follows from Proposition 3.1 (and its proof) that

$$(-\hbar \varphi(\delta_i, \delta_j))_{i,j=1,2} = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}, \quad (\varphi(\delta_i, \delta_j))_{i,j=1,2}^{-1} = \hbar \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Set $\delta_1^* := \hbar(-2\delta_1 + \delta_2)$ and $\delta_2^* = \hbar\delta_1$. From the relations of Drinfeld generators it follows that A_{\hbar} (resp. B_{\hbar}) has a topological basis $\delta_1^{m_1} \delta_2^{m_2} E(f)$ (resp. $(\hbar^{-1} \delta_1^*)^{m_1} (\hbar^{-1} \delta_2^*)^{m_2} F(f)$) where $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ and $f \in \Gamma$. By Proposition 4.1, these two bases are orthogonal with respect to φ . The universal R -matrix \mathcal{R} is then the associated Casimir element:

$$(5.12) \quad \mathcal{R} = \mathcal{K} \mathcal{R}_- \mathcal{R}_0 \mathcal{R}_+ \in A_{\hbar} \widehat{\otimes} B_{\hbar},$$

$$(5.13) \quad \mathcal{K} = e^{\delta_1 \otimes \delta_1^* + \delta_2 \otimes \delta_2^*} = q^{\delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1 - 2\delta_1 \otimes \delta_1},$$

$$(5.14) \quad \mathcal{R}_- = \prod_{b \in \mathcal{B}_-}^{\rightarrow} \left(1 - \frac{b \otimes b^-}{\varphi(b, b^-)} \right) = \prod_{s=\infty}^1 \left(1 + \frac{(\phi_0^+)^{-1} F_s \otimes (\phi_0^-)^{-1} E_{-s}}{q - q^{-1}} \right),$$

$$(5.15) \quad \mathcal{R}_+ = \prod_{b \in \mathcal{B}_+}^{\rightarrow} \left(1 - \frac{b \otimes b^-}{\varphi(b, b^-)} \right) = \prod_{n=0}^{\infty} \left(1 + \frac{E_n \otimes F_{-n}}{q^{-1} - q} \right),$$

$$(5.16) \quad \mathcal{R}_0 = \exp\left(\sum_{b \in \mathcal{B}_0} \frac{b \otimes b^-}{\varphi(b, b^-)}\right) = \exp\left((q - q^{-1}) \sum_{s>0} \frac{s}{[s]} (q^{-s} h_s \otimes C_{-s} + q^s C_s \otimes h_{-s})\right).$$

Remark 5.1. By the quantum double construction, we have: for $x \in U$

$$(\text{Id}_A \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (\Delta \otimes \text{Id}_B)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad \mathcal{R} \Delta(x) = \Delta^{\text{cop}}(x) \mathcal{R}.$$

Let $\mathcal{R}(z), \mathcal{R}_{\pm,0}(z)$ be obtained from $\mathcal{R}, \mathcal{R}_{\pm,0}$ by replacing the F_s, E_n, h_s, C_s in the last three equations with the $F_s z^s, E_n z^n, h_s z^s, C_s z^s$ respectively. So $\mathcal{R}_{\pm,0}(z) \in U^{\otimes 2}[[z]]$. As a first application, let us deduce the Perk-Schultz matrix $R(z, w)$ in Equation (2.1) from \mathcal{R} . There is a natural representation ρ of U_h on the vector superspace \mathbf{V} [Zh2, §4.4]:

$$(\rho(s_{ij}(w)))_{1 \leq i, j \leq 2} = \begin{pmatrix} (q - q^{-1}w)E_{11} + (1 - w)E_{22} & (q - q^{-1})E_{12} \\ (q - q^{-1})wE_{21} & (1 - w)E_{11} + (q^{-1} - qw)E_{22} \end{pmatrix}.$$

From the Gauss decomposition we can deduce the action of the Drinfeld generators:

$$\rho(E_n) = q^{-2n-1}(q - q^{-1})E_{12}, \quad \rho(F_n) = q^{-2n+1}(q^{-1} - q)E_{21}, \quad \rho(C_s) = -\frac{q^{-s}[s]}{s},$$

for $s \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{Z}$. Now let us compute the following $R(z) \in \text{End} \mathbf{V}^{\otimes 2}[[z]]$

$$\begin{aligned} R(z) &:= \rho^{\otimes 2}(\mathcal{R}(z)) = \rho^{\otimes 2}(\mathcal{K})R_-(z)R_0(z)R_+(z), \\ \rho^{\otimes 2}(\mathcal{K}) &= \rho^{\otimes 2}(q^{-(\delta_1 - \delta_2) \otimes (\delta_1 - \delta_2) - \delta_1 \otimes \delta_1 + \delta_2 \otimes \delta_2}) = q^{-1 - E_{11} \otimes E_{11} + E_{22} \otimes E_{22}}, \\ R_-(z) &= \rho^{\otimes 2}\left(\prod_{s>0} \left(1 + \frac{(\phi_0^+)^{-1} z^s F_s \otimes (\phi_0^-)^{-1} E_{-s}}{q - q^{-1}}\right)\right) \\ &= 1 - \sum_{s>0} \frac{z^s}{q^{-1} - q} (q^{-1} - q)(q - q^{-1})E_{21} \otimes E_{12} = 1 - \frac{(q - q^{-1})z}{1 - z} E_{21} \otimes E_{12}, \\ R_0(z) &= \rho^{\otimes 2} \exp((q - q^{-1}) \sum_{s>0} \frac{sz^s}{[s]} (q^{-s} h_s \otimes C_{-s} + q^s C_s \otimes h_{-s})) \\ &= \left(\frac{1}{1 - q^{-2}z} E_{11} + \frac{1}{1 - z} E_{22}\right) \otimes ((1 - q^2 z)E_{11} + (1 - z)E_{22}), \\ R_+(z) &= \rho^{\otimes 2}\left(\prod_{n \geq 0} \left(1 + \frac{z^n E_n \otimes F_{-n}}{q^{-1} - q}\right)\right) = 1 + \frac{q - q^{-1}}{1 - z} E_{12} \otimes E_{21}. \end{aligned}$$

It turns out that $R(z) = \frac{R(z, w)}{q^{-1}z - qw}|_{w=1}$.

As another application, let us deduce the Baxter polynomiality for U in the spirit of Frenkel-Hernandez [FH]. For $a \in \mathbb{C}^\times$, there is a representation π_a of A on \mathbf{V} defined by:

$$(\pi_a(s_{ij}(z)))_{1 \leq i, j \leq 2} = \begin{pmatrix} \frac{1}{1 - za} E_{11} + \frac{q^{-1}}{1 - za} E_{22} & \frac{q^{-1} - q}{1 - za} E_{12} \\ \frac{-za}{1 - za} E_{21} & E_{11} + \frac{q^{-1} - zaq}{1 - za} E_{22} \end{pmatrix}.$$

Based on the Gauss decomposition, we get: for $s > 0, n \geq 0$

$$\begin{aligned} \pi_a(\delta_1) &= \pi_a(\delta_2) = -E_{22}, \quad \pi_a(h_s) = \frac{a^s}{s(q - q^{-1})} \text{Id}_{\mathbf{V}} = -\pi_a(C_s), \\ \pi_a(E_n) &= \delta_{n0}(q^{-1} - q)a^n E_{12}, \quad \pi_a(F_s) = \delta_{s1}a^s E_{21}. \end{aligned}$$

For $c, d \in \mathbb{C}^\times$ with $c \neq \pm 1$, there is a representation $\rho_{c,d}$ of U on \mathbf{V} :

$$(\pi_{c,d}(s_{ij}(z)))_{1 \leq i, j \leq 2} = \begin{pmatrix} c \frac{1 - zd}{1 - zdc^2} E_{11} + c \frac{q^{-1} - zdq}{1 - zdc^2} E_{22} & c \frac{(q^{-1} - q)(dc^2 - d)}{1 - zdc^2} E_{12} \\ \frac{-z}{1 - zdc^2} E_{21} & E_{11} + \frac{q^{-1} - zdc^2 q}{1 - zdc^2} E_{22} \end{pmatrix}$$

and $\pi_{c,d}(t_{ij}(z)) = -z^{-1}d^{-1}c^{-2}(1 - zdc^2)(1 - z^{-1}d^{-1}c^{-2})^{-1}\pi_{c,d}(s_{ij}(z))$. Similarly

$$\begin{aligned} \pi_{c,d}(\phi_0^\pm) &= c^{\mp 1}, \quad \pi_{c,d}(h_s) = \frac{d^s}{s} \left(\frac{c^{2s} - 1}{q - q^{-1}} E_{11} + \frac{c^{2s} - q^{2s}}{q - q^{-1}} E_{22} \right), \quad \pi_{c,d}(C_s) = -\frac{d^s}{s} \frac{c^{2s} - 1}{q - q^{-1}} \text{Id}_{\mathbf{V}}, \\ \pi_{c,d}(E_n) &= (q^{-1} - q)(dc^2 - d)d^n E_{12}, \quad \pi_{c,d}(F_n) = d^{-1}c^{-1}d^n E_{21}. \end{aligned}$$

A straightforward calculation indicates that

$$\begin{aligned} (\pi_a \otimes \pi_{c,d})(\mathcal{R}(z)) &= f_{c,d}(az)R_{c,d}(az) \in \text{End}(\mathbf{V}^{\otimes 2})[[z]], \\ f_{c,d}(z) &= \exp\left(\sum_{s>0} \frac{(q^s + q^{-s})(c^{-2s} - 1)d^{-s}}{s(q^s - q^{-s})} z^s\right), \\ R_{c,d}(z) &= E_{11} \otimes E_{11} + \frac{1}{1 - d^{-1}z} E_{11} \otimes E_{22} + \frac{c - d^{-1}c^{-1}z}{1 - d^{-1}z} E_{22} \otimes E_{11} \\ &\quad + \frac{c}{1 - d^{-1}z} E_{22} \otimes E_{22} + \frac{d^{-1}c^{-1}}{1 - d^{-1}z} E_{12} \otimes E_{21} + \frac{(1 - c^2)z}{1 - d^{-1}z} E_{21} \otimes E_{12}. \end{aligned}$$

Now let $c_j, d_j \in \mathbb{C}^\times$ be such that $c_j^2 \neq 1$ for $1 \leq j \leq n$. Let $\pi_{\underline{c}, \underline{d}}$ be the tensor product representation $\otimes_{j=1}^n \pi_{c_j, d_j}$ of U on $W := \mathbf{V}^{\otimes n}$. For $0 \leq m \leq n$, let W_m be the subspace of W spanned by the tensors $v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n}$ containing exactly m v_1 's. Write

$$(\pi_a \otimes \pi_{\underline{c}, \underline{d}})(\mathcal{R}(z)) = E_{11} \otimes A_{11}(az) + E_{22} \otimes A_{22}(az) + E_{12} \otimes A_{12}(az) + E_{21} \otimes A_{21}(az)$$

with $A_{ij}(z) \in \text{End}W[[z]]$. The W_m are stable by the $A_{ii}(z)$. By Remark 5.1:

Corollary 5.2. $\left(\prod_{j=1}^n \frac{1 - d_j^{-1}z}{f_{c_j, d_j}(z)}\right) \times A_{ii}(z)|_{W_m}$ is a polynomial in z of degree m for $i = 1, 2$.

This resembles [FH, Theorem 5.9] (if we regard π_a as the positive prefundamental module $R_{i,a}^+$ in *loc. cit.*). By definition $A_{11}(z) = \pi_{\underline{c}, \underline{d}}(T(z))$ where

$$T(z) := \exp\left(\sum_{s>0} \frac{z^s(q^{-s}C_{-s} - q^s h_{-s})}{[s]}\right) \in U[[z]].$$

By [Zh2], all finite-dimensional simple U -modules are tensor products of the $\pi_{c,d}$ with one-dimensional modules. The corollary above also indicates the polynomial action of $T(z)$ on tensor products of finite-dimensional simple U -modules. (Compare [FH, Theorem 5.13].)

We end this section with the following Drinfeld new coproduct on U . Let $\Delta_z : U \rightarrow U^{\otimes 2}[z, z^{-1}]$ be obtained from Δ by replacing $v \otimes w \in U^{\otimes 2}$ with $z^n v \otimes w$ whenever $|v|_{\mathbb{Z}} = n$. Define $\Delta_z^{(D)}(x) := \mathcal{R}_+(z)\Delta_z(x)\mathcal{R}_+(z)^{-1} \in U^{\otimes 2}((z))$ for $x \in U$. By Proposition 2.1:

$$\begin{aligned} \Delta_z^{(D)}(h_s) &= 1 \otimes h_s + z^s h_s \otimes 1, \quad \Delta_z^{(D)}(C_s) = 1 \otimes C_s + z^s C_s \otimes 1, \\ \Delta_z^{(D)}(E_n) &= 1 \otimes E_n + \sum_{k \geq 0} z^{n+k} E_{n+k} \otimes \phi_{-k}^-, \quad \Delta_z^{(D)}(F_n) = z^n F_n \otimes 1 + \sum_{k \geq 0} z^k \phi_k^+ \otimes F_{n-k}. \end{aligned}$$

Similar Drinfeld new coproduct has been obtained in [Zy] by different methods.

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REFERENCES

- [ACFR] D. Arnaudon, N. Crampé, L. Frappat and E. Ragoucy, *Super Yangian $Y(\mathfrak{osp}(1|2))$ and the universal R -matrix of its quantum double*, Commun. Math. Phys. **240** (2003), 31-51.
- [CWWX] J. Cai, S. Wang, K. Wu and C. Xiong, *Universal \mathcal{R} -matrix of the super Yangian double $DY(\mathfrak{gl}(1|1))$* , Commun. Theo. Phys. **29** (1998), 173-176.
- [CWWZ] J. Cai, S. Wang, K. Wu and W. Zhao, *Drinfel'd realization of quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}(1|1)})$* , J. Phys. A: Math. Gen. **31**(1998), 1989-1994.
- [Da1] I. Damiani, *La \mathcal{R} -matrice pour les algèbres quantiques de type affine non tordu*, Ann. Sci. Ecole Norm. Sup. **31** (1998), 493-523.
- [Da2] I. Damiani, *The R -matrix for the (twisted) quantum affine algebras*, Representations and quantizations (Shanghai 1998), 89-144, China High. Educ. Press, Beijing, 2000, arXiv:1111.4085.
- [FH] E. Frenkel and D. Hernandez, *Baxter's relations and spectra of quantum integrable models*, Duke Math. J. to appear, Preprint arXiv:1308.3444.
- [FR] E. Frenkel and N. Reshetikhin, *The q -character of representations of quantum affine algebras and deformations of \mathcal{W} -algebras*, Recent Developments in Quantum Affine Algebras and related topics, Cont. Math. **248** (1999), 163-205.
- [Ga] R. Gade, *Universal R -matrix and graded Hopf algebra structure of $U_q(\widehat{\mathfrak{gl}(2|2)})$* , J. Phys. A: Math. Gen. **31** (1998), 4909-4925.
- [IZ] I. Ip and A. Zeitlin, *Q -operator and fusion relations for $C_q^{(2)}(2)$* , Lett. Math. Phys. **104** (2014), 1019-1043.
- [Ka] M. Kashiwara, *On level-zero representations of quantized affine algebras*, Duke Math. J. **112** (2002), 117-175.
- [KKK] S. Kang, M. Kashiwara and M. Kim, *Symmetric quiver Hecke algebras and R -matrices of quantum affine algebras*, Preprint arXiv:1304.0323.
- [KT1] S. Khoroshkin and V. Tolstoy, *Universal R -matrix for quantum (super)algebras*, Commun. Math. Phys. **141** (1991), 599-617.
- [KT2] S. Khoroshkin and V. Tolstoy, *The universal R -matrix for quantum untwisted affine Lie algebras*, Funct. Anal. Appl. **26** (1992), 69-71.
- [KT3] S. Khoroshkin and V. Tolstoy, *Yangian double*, Lett. Math. Phys. **36** (1996), 373-402.
- [RS] A. Rej and F. Spill, *The Yangian of $\mathfrak{sl}(m|n)$ and its quantum R -matrices*, JHEP **05** (2011), 012.
- [Zh1] H. Zhang, *Representations of quantum affine superalgebras*, Math. Z. **278** (2014), 663-703.
- [Zh2] H. Zhang, *RTT realization of quantum affine superalgebras and tensor products*, Preprint arXiv:1407.7001.
- [Zh3] H. Zhang, *Asymptotic representations of quantum affine superalgebras*, Preprint arXiv:1410.0837.
- [Zy] Y. Zhang, *Comments on the Drinfeld realization of quantum affine superalgebra $U_q(\mathfrak{gl}(m|n)^{(1)})$ and its Hopf algebra structure*, J. Phys. A: Math. Gen. **30** (1997), 8325-8335.

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